LEARNING CALCULUS WITH SUPERCALCULATORS

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A CAS-active introductory calculus course was trialled with three year 11 classes of students who each had a TI-92 calculator. Access to the CAS enabled the students to perform "year 12" differentiations with the same level of success as year 12 students, but without affecting their conceptual learning or their development of by-hand skills. The use of the CAS was influenced by teachers' cognitive "privileging". Some observations are made on the suitability of the TI-92 for such a course.

INTRODUCTION

Computer algebra systems (CAS), such as Maple, Mathematica and Derive, have been used by mathematicians for at least 20 years, but it is only in recent years that advances in technology have enabled CAS to be incorporated into graphics calculators—producing so called 'supercalculators', such as the Texas Instruments TI-92 calculator. The changes currently being incorporated into the secondary mathematics curriculum as a result of the availability of graphing calculators are insignificant compared with the potential impact of supercalculators. This is illustrated, for example, by McCrae's (1996) finding that, whereas the availability of a graphing calculator would impact on only 6% of the 1994 Victorian Certificate of Education Specialist Mathematics paper, about 79% would be affected by the availability of a CAS. Similarly, Shumway (1989) found that as much as 90% of the exercises in most USA mathematics texts could be computed directly by a CAS.

However, as those mathematicians who have most strongly influenced the development of supercalculators from a pedagogic perspective have observed:

> We have much to learn about CAS in the teaching and learning of mathematics. We need to understand how CAS and paper and pencil procedures can and should co-exist. We need to learn what paper and pencil skills are necessary to use CAS effectively. We need to determine what mental computations in algebra and calculus are important for student learning. We need to learn how to test with CAS because there is much evidence that the only thing that really influences what our students learn is 'that what is tested'. We must learn how to teach the next level above 'doing' or practising procedures—thinking about mathematics. (Waits, Demana & Kutzler, 1997, p.5)

Tynan and Asp (1998) investigated the impact of the availability of a TI-92 calculator on the performance of year 9 students on a range of symbol manipulation tasks. They found that, compared to a class that used a graphing calculator with no in-built symbol manipulation features, the TI-92 students were more inclined to persist with algebraic methods for equation solving tasks. They also found that there was no significant difference between the two classes in their ability to perform by-hand algebraic manipulations.

Just as the transition from primary mathematics to junior secondary mathematics can be characterised by the introduction of algebra, the transition to senior mathematics can be characterised by the introduction of calculus. Hillel (1993, p. 29) has observed that CAS has the potential for affecting a major reorientation of the teaching of calculus so as to "focus on concepts, problem-solving skills and general investigative activities rather than manipulative skills", but adds that the "real tough" pedagogical question is "how much of the manipulative aspect can be eliminated while still sustaining conceptual learning"?

Keller and Russell (1997) note that despite the fact that several studies concerned with using a CAS in calculus have found improved understanding of the concepts of calculus with no significant loss of by-hand manipulative skill, concern remains that decreased emphasis on skills will lead to decreased ability to solve calculus problems symbolically. A major reason for the lack of impact of the research on practice is that in all of these studies either students' access to CAS was limited to computer laboratory sessions or the relevant examinations had to be done without access to CAS (or both)—meaning that there was still a strong incentive for students to develop their by-hand skills. The advent of supercalculators enables research to be conducted without these restrictions on CAS accessibility in what we might term a CAS-active environment.

In their study of first-year engineering students, Keller and Russell found that unrestricted access to a CAS actually empowered students to solve calculus problems symbolically. This paper reports some of the findings of a study that involved the development and piloting of a CAS-active introductory calculus course for year 11 students. The main objectives of the study were to determine whether the availability of the CAS enabled the students to solve more advanced problems than those normally encountered in such an introductory course, without adversely affecting their level of conceptual learning, and to investigate the effect of different teaching approaches in a CAS-active environment. Also, the general suitability of the TI-92 (the CAS platform involved in the study) for use in such a course was documented.

METHODOLOGY

During 1998, a CAS-active introductory calculus course was developed and then piloted in three year 11 classes in Victoria, two from a girl's grammar school and one from a State high school. Each student was loaned a TI-92 calculator for use in class and at home. The students were already familiar with the TI-83 graphical calculator and used the TI-92 in a two-week unit on circular functions prior to the commencement of the calculus unit.

Students from both schools were in the lowest ability mathematics classes in their year level and the classes were similar in size: there were 16 girls in one of the grammar school classes and 24 in the other, whilst there were 19 students (13 girls and 6 boys) in the high school class. The three teachers were experienced in teaching year 11 and 12 mathematics with the TI-83 graphical calculator but inexperienced in using the TI-92 with its additional CAS capability. The three teachers had quite different teaching styles.

Twenty lessons were devoted to teaching differentiation with a much stronger emphasis on conceptual understanding than had been the case in previous years. Introductory rates of change were explored using real data with a CBR linked to the TI-92. A program was written to demonstrate dynamically the connection between the tangent at a given point and a series of moving secant lines drawn very close to the point. The derivative was described in numerical, graphical and algebraic representations. Students were encouraged to discover the rules for differentiating polynomials by induction and given the opportunity to apply their knowledge in a wide variety of contexts.

The overall scope of the course was identical to that normally covered in the same period and included some topics that officially are part of the year 12 course: deducing the graph of the gradient function from the graph of a function, derivatives of rational powers, and the chain and product rules. The extra time devoted to concept development and applications was offset by less time spent on practising differentiation techniques, with the teachers being asked not to place the usual emphasis on the development of by-hand skills. (A two-week block had been set aside after the completion of the course for student's by-hand skills to be brought up to scratch if this proved necessary.)

The students were given an informal test about half way through the seven-week teaching period. In most weeks they completed a brief questionnaire and a few short questions designed to monitor their progress and log sheets which gave them the opportunity to describe their feelings about using the TI-92 to learn calculus. Three tests were administered in week 8 and interviews were conducted with 17 selected students the following week. The teachers completed journals and a questionnaire and participated in a taped interview and regular informal discussions. A Research Assistant observed approximately half of the lessons taught and completed a journal after each lesson.

The students were able to use their TI-92 in the first two of the three final tests, but were only permitted to use a (non-graphics) scientific calculator in the third test. At the end of each question part on Tests 1 and 2, a YES or NO box had to be checked to indicate whether the TI-92 had been used to help answer the item. The tests were not typical year 11 calculus tests. They were designed to test the students' conceptual understanding and whether they could do relatively advanced differentiations. Many of the items on the tests came from past year 12 examination papers.

Test 1 consisted of 14 multiple-choice questions and several items requiring short answer responses. Test 2 consisted of items requiring extended responses and Test 3 comprised a selection of items from Tests 1 and 2. For comparison, the multiple-choice questions were also given to another year 11 class (15 students), of comparable mathematical ability, and one mainstream year 12 class (21 students) from the high school. The year 11 class had just completed the normal introductory calculus course and the year 12 class had completed the calculus part of their course; neither of these 2 classes had access to a CAS for the test. This paper looks at the comparative performances of the five classes on the 14 multiple-choice questions.

RESULTS AND DISCUSSION

Table 1 shows the mean scores of each class for 13 of the multiple-choice questions (one question has been excluded from the analysis because not all classes covered the topic). Each question was awarded 0 (incorrect) or 2 (correct), so that the maximum possible mark was 26. Classes A, B and C are the year 11 CAS classes, class D is the year 11 non-CAS class and class E is the year 12 class. The values of N are given throughout this discussion, since not all students were present for each test.

Table 1
Mean Scores for Multiple-Choice Questions by Class

	Class A	Class B	Class C	Class D	Class E
N	15	19	16	15	21
Mean	14.0	14.1	14.3	8.4	17.0

The means for the three CAS classes are much higher than the non-CAS year 11 mean and although they aren't as high as the year 12 class mean, they are quite pleasing in view of the difficulty of the questions. It is likely that the CAS students must have used their TI-92s to advantage and this will become clearer in the following analysis.

Twelve of the 13 questions can be readily classified as either being a *core* item or a *symbolic* item (see Kendal & Stacey, in press). Core items have high conceptual and low procedural demands; there is no advantage in using a CAS in such questions. One example was question 9, which gave the graph of an unknown function and required the graph of its derivative to be selected from five alternatives. Symbolic items have high demands on algebraic procedures and low conceptual demands; use of a CAS can be a distinct advantage in answering such questions. One example was question 3: "If $y = (4x^2+9)^7$, then dy/dx is equal to ..." Access to a CAS makes this question almost trivial.

The first 6 multiple-choice questions were symbolic items and these questions were also included on Test 3—the test given to the CAS students about a week after Test 1 and for which the students were not allowed to use their TI-92s. Table 2 shows the mean scores of each class for these questions, including the second attempts for classes A, B and C. Only those students who sat for both Tests 1 and 3 were included in the calculations for classes A, B and C (N=15 for class A, 16 for class B and 13 for class C).

Table 2
Mean Scores for Symbolic Items by Class

Question:	1	2	3	4	5	6	Total
Class A—with CAS	1.87	1.73	0.80	1.60	1.33	0.80	8.1
Class A—without CAS	1.87	0.40	0	0.13	0.80	0.80	4.0
Class B—with CAS	2	1.63	1.00	1.50	1.00	1.25	8.4
Class B—without CAS	1.75	0.63	0.50	0.75	1.00	0.88	5.5
Class C—with CAS	2	1.54	0.92	1.69	0.62	0.77	7.5
Class C—without CAS	1.54	0.15	0	0	0.77	0.77	3.2
—with CAS mean	*1.95*	*1.64*	*0.91*	*1.59*	*1.00*	*0.95*	*8.0*
—without CAS mean	*1.73*	*0.41*	*0.18*	*0.32*	*0.86*	*0.82*	*4.3*
Class D (year 11)	1.73	0.27	0.13	0.13	0.53	0.67	3.5
Class E (year 12)	2	0.86	1.05	0.95	1.43	1.43	7.7

The "with CAS" class mean totals are higher than the year 12 class mean total for two of the classes (A and B), and comparable for the other class (C). Notwithstanding the variations between questions, this indicates that access to a CAS enables year 11 students to solve symbolic items of "year 12 standard" with the same degree of success as year 12 students. Also, the "without CAS" class mean totals are higher than the non-CAS year 11 class mean total for two of the classes (A and B) and slightly lower for class C. This is in agreement with the results for question 1, the only one of the questions that was clearly of year 11 standard (a cubic differentiation), and indicates that the CAS students did not suffer unduly from less emphasis on by-hand skills in their course. The variations in results between CAS classes will be discussed later.

Table 3 shows the mean scores for each of the five classes on 5 questions that were clearly core items. (One "clearly core" question has been excluded from the analysis because, arguably, it has two answers.) Values of N are as given in Table 1. The scores indicate that the conceptual understanding of the CAS students was generally better than that of the non-CAS year 11 students, but not as good as the year 12 students. The results are more favourable to the CAS students if question 7, on which each CAS class performed particularly badly, is ignored. This question required students to calculate the slope of a curve at two points by calculating the gradients of the tangents at those points. Analysis of students' responses revealed that they were just as likely to calculate the slope of the secant through the two points or to choose the relevant x- or y-coordinate as the slope of the curve. This aspect of the course obviously needs re-examination before further trialling.

Table 3
Mean Scores for Core Items by Class

Question:	7	9	10	11	14	Total	Total (without q7)
Class A	0.27	0.67	0.93	1.47	0.80	4.1	3.9
Class B	0.21	1.05	1.16	1.16	0.74	4.3	4.1
Class C	0.38	1.00	1.00	1.75	1.13	5.3	4.9
A, B, C mean	*0.28*	*0.92*	*1.04*	*1.44*	*0.88*	*4.6*	*4.3*
Class D	0.67	0.53	0.40	1.73	0.67	4.0	3.3
Class E	1.62	1.05	1.52	1.91	0.86	6.9	5.3

Comparing the CAS classes to each other, class C students demonstrated the highest level of conceptual understanding. To help determine which class made best use of the CAS,

Table 4 further analyses the responses to the 6 symbolic items referred to in Table 2. For each class, it shows the percentage of total student responses in which students reported using their TI-92, and the "overall", "with CAS" and "without CAS" success rates. It indicates that greater success does not necessarily follow from higher CAS use: class A students chose to use their TI-92 almost twice as often as did class B and C students, but were less successful overall than class B students. On the other hand, it is likely that class B and C students would have profited from using their CAS more often.

Table 4
Percentage of Items Correct by CAS use and Class

	Class A (N=15)	Class B (N=19)	Class C (N=16)
Total no. of items	90	114	96
% correctly answered	67.8	69.3	60.4
% for which CAS used	64.4	36.8	35.4
% correct when CAS used	79.3	95.2	82.4
% correct otherwise	46.9	54.2	48.4

Kendal and Stacey (in press) extend this analysis to the remaining Test 1 and Test 2 questions, some of which are *options* items: they can be solved graphically or algebraically. They also look at the incidence of conceptual and procedural errors and compare the characteristics of the three teachers involved. Kendal and Stacey's summary of the relative behaviours of the three CAS classes is reproduced as Table 5.

Table 5
Summary of the Behaviours of the Three CAS Classes (Kendal & Stacey, in press)

Behaviour	Class A	Class B	Class C
Use of calculator	most frequent	least frequent	frequent
Decision to use calculator	too frequent	discriminating	discriminating
Preferred approach	algebra by CAS	algebra by hand	graphical
Algebra proficiency	moderate by hand	higher by hand	lower by hand
Graphical skills	lower	moderate	higher
Procedural competence	good	good	good
Conceptual understanding	lower	moderate	higher

Berger (1998) describes Wertsch's concept of "privileging" as "the social setting and values which may elevate one form of mental functioning over another and in this way privilege a particular form of mental operation such as algebraic or graphical reasoning" (p. 19). Kendal and Stacey attribute the differences between the behaviours of the three classes to the different privileging of the teachers involved. According to them, Teacher A privileged technological and algebraic approaches, Teacher B privileged conceptual understanding and by-hand algebraic approaches and Teacher C privileged graphical approaches and conceptual understanding.

SUITABILITY OF THE TI-92

The students in the CAS classes were very enthusiastic about the ability of the TI-92 to "do both graphs and algebra". Otherwise, the feature that they most often regarded as "great" was the ability to view and recall previous work. The most frustrating feature was the TI-92's general complexity: "too many buttons and not knowing what they all do", "there are too many pull down menus and they are too complicated". It would have helped if each student could have been given a manual, but a better user interface is what is really needed.

The fact that (identical, but) different factorisations can arise depending on the mode setting often led to confusion: for example, factor($2x^2 + 3x - 2$) gives $(x+2)(2x-1)$ in auto or exact mode but $2.(x-.5)(x+2.)$ in approximate mode. However, as noted by Tynan and Asp (1998, p. 626), probably the most frustrating feature of the TI-92 is the way in which its autosimplify feature alters the form of expressions. Any reader who doubts this is invited to try and derive the formula for the solution of the general quadratic equation in its usual form.

CONCLUSION

Before long, supercalculators will cost no more than non-CAS graphing calculators do at present. Every kid in school will have one and we must be prepared. This study indicates that, if calculus is introduced with a strong emphasis on conceptual understanding, then year 11 students can use a CAS to do "year 12" problems. It will no longer be necessary for students to spend so much time learning differentiation and antidifferentiation techniques; many common assessment items will be trivialised. We must act now to revise mathematics syllabuses and assessment regimes accordingly.

Further studies are needed to investigate the impact of CAS on the learning of calculus. Perhaps more importantly though, its impact on the learning of algebra needs thorough study. Indications are that, paradoxically, a strong algebra facility is needed to get the best out of a CAS. Each CAS has a mind of its own and weak points that need to be overcome.

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